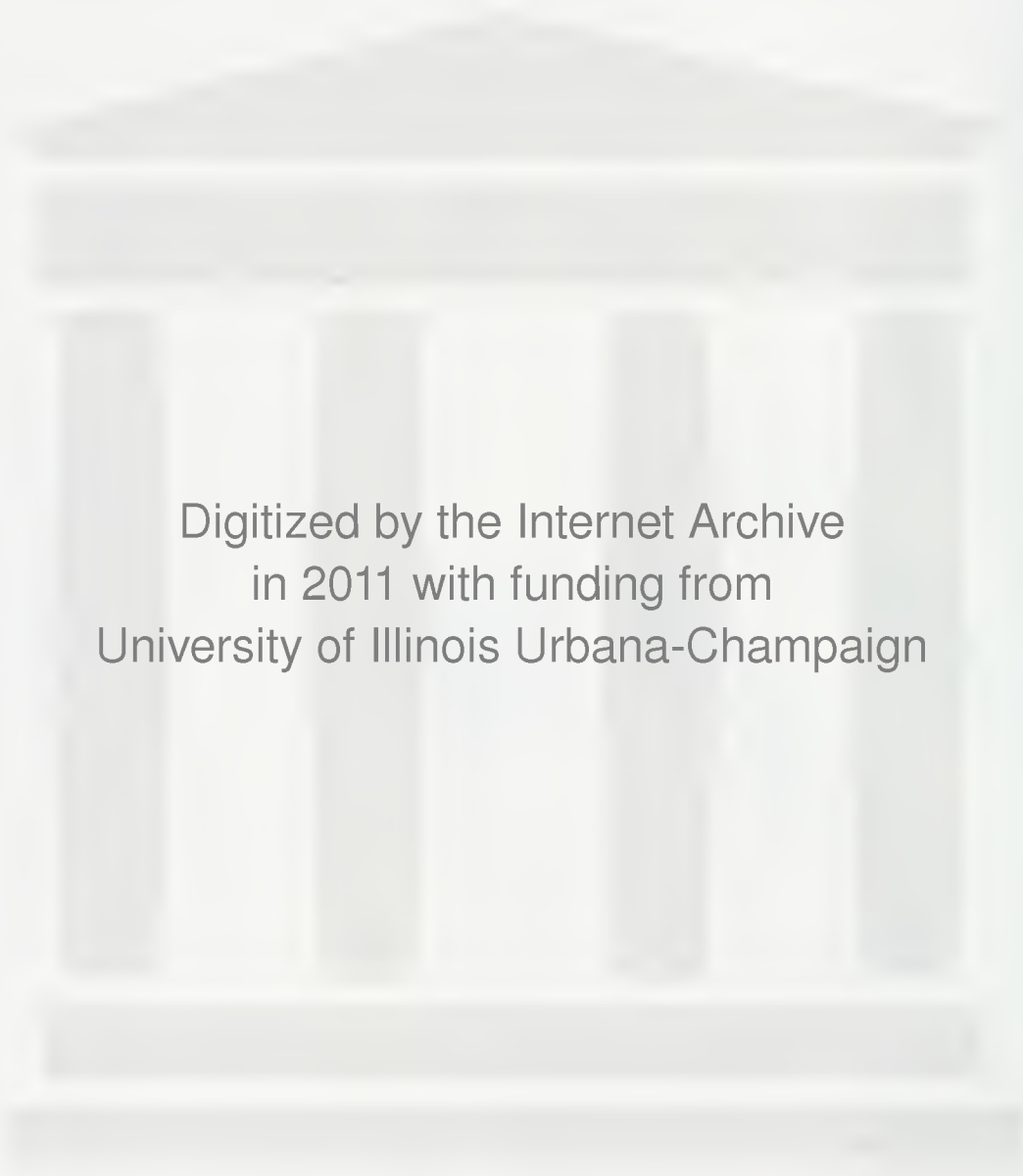
UNIVERSITY OF
ILLINOIS LIBRARY
AT URBANA-CHAMPAIGN
BOOKSTACKS


Digitized by the Internet Archive
in 2011 with funding from
University of Illinois Urbana-Champaign

http://www.archive.org/details/transitionfuncti348evan

Faculty Working Papers

TRANSITION FUNCTIONS FOR EVENT PROCESSES

Richard V. Evans

#343

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

FACULTY WORKING PAPERS

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

October 20, 1976

TRANSITION FUNCTIONS FOR EVENT PROCESSES

Richard V. Evans

#343

Transition Functions for Event Processes

Richard V. Evans
Professor, Business Administration

Summary: A general continuous time congestion process is defined on a finite number of independent sequences of i,i,d random variables and routing probabilities. After examining transitions in a small time interval, attention is focused on an iterative calculation which is interpreted as a discrete time Markov process.

From a qualitative point of view a single channel queue should provide an approximation of the congestion in a system in which homogeneous customers are processed by an organization of specialized servers. On the other hand, investigating simple processing systems can expand understanding of simple queues. When a system can work on several jobs simultaneously, the times between service completions during a busy period can be considered as service times for successive customers in a single server system. These times are likely to be correlated. With this interpretation a processing system model provides an interesting and potentially useful generalization of the usual single server models. The relationship between single server and processing systems can be made quantitative through the study of a class of network models.

Event Structure

The single channel queue is modeled by a stochastic process based on the arrival and service events. The generalization discussed here can still be defined in terms of events which cause discontinuities in the congestion process. The innovation is that here there will be a finite number, k*, of event types. Each event of each type is proceeded by an interval of time. These times form k* independent sequences of i,i,d random variables. As in the single channel system, the occurance of an event may start any number of intervals. Again an arrival always starts a new interarrival interval so that there is always at least one interval in process. In these systems it is desireable to introduce a new possible effect of an event. This is that an event of one type may in appropriate circumstances cause an interruption

in the interval associated with another type of event. In the language of priority systems the interruption may result in the resumption of the interval at some later time or it may cause the repetition of that part which has already elapsed. Although more stringent than necessary, a convenient and useful assumption is that the occurance of an event cannot shorten any of the intervals which are in process except by cancelling the coming event entirely. Under this assumption the number of events of each type i in any interval of time τ is not greater than one plus the minimum number of intervals of type i whose sum is greater than τ. Summing over the k* types of intervals provides an upper bound on the number of events in τ. To ensure that there are no technical difficulties in constructing probability relationships, it is desirable to insure that this bound is finite with probability 1. An assumption which guarantees this is that there is some $\varepsilon > 0$ such that the distribution function of the i th interval type satisfies $F_i(\varepsilon) < \alpha < 1$.

A further assumption, which helps to simplify the construction of these processes and is not very restrictive, guarantees that for a small Δt at most one event occurs. If an interval of each type is in process at time t then the probability of two or more events occurring in Δt is

$$p(\Delta t) = 1 - \prod_{i=1}^{k^*} (1-H_i(\xi_i)) - \sum_{i=1}^{k^*} [H_i(\xi_i) \prod_{\substack{j=1 \\ j\neq i}}^{k^*} (1 - H_j(\xi_j))]$$

where ξ_i is the time since the start of i th interval and $H_i(\xi_i)$ is the conditional probability that an interval of type i will terminate in an interval of length Δt given that it has not terminated in $[0,\xi_i]$

$$H_i(\xi_i) = \frac{dF_i(\xi_i)}{1-F_i(\xi_i)}$$

where $dF_i(\xi_i)$ is the probability that an interval of type i will end in

$[\xi_i, \xi+\Delta t]$. Assuming that the interval distributions have continuous density functions implies $H_i(\xi_i)=h_i(\xi_i)\Delta t$ and the probability of two or more events becomes $o(\Delta t)$. Thus the probability that the congestion process has at most one event in any small interval of time Δt can be made as close to 1 as one wishes by choosing Δt sufficiently small. The probability of an event of type i is $h_i(\xi_i)\Delta t$ to this same level of approximation. These same conclusions hold for the processess which are in operation when there are fewer than k* of them. Since the arrival process always operates there will always be at least one interval in process at any time.

These assumptions and conclusions should not be a surprise. As with the special case of 2 event systems, these processes depend on event sequences. With probability 1 only a finite number of events occur in any finite interval and at most one occurs in any interval of time which is sufficiently small. For the special case of k*=2, examining the congestion only at the occurrance times of one type of event provides important simpliciations in analysis. As k* increases the significance of semi-Markov process analysis decreases dramatically.

Routing

In the single channel queue the changes which occur at arrival and service completion events are easy to describe. This is not the case in more general processing systems. Jobs do not remain homogeneous in such systems but become distinguished as various service tasks are performed upon them. Thus general systems automatically acquire some of the characteristics of priority systems. One concept of a general congestion process is that jobs change priority after each service operation and each free server starts work on the job with the highest priority among those on which he can work. This also raises the possibility of jobs preempting servers which is the reason for considering interruption of the intervals to the next event of various types.

For this discussion only a finite number of priority classes will be allowed. Moreover, any event can change the priority classification of only one job. For effective analysis, the change in priorities must be Markovian i.e. depend only on the congestion at the time of a change. The variety of change processes which can be considered depends on how the states of the congestion process are defined. If the lengths of the intervals in progress are included, an interesting possibility is that priority is determined by the length of time spent in the service operation just completed. This is used to classify short and long jobs in computing systems. A job which requires an i.o. device before finishing a quantum of cpu time is considered short while one which does not is classified as long. Such a classification proceedure does not complicate the models discussed here since to be Markovian the congestion process requires supplementary variables describing the time since the start of the current interarrival interval and all services currently in progress. Thus appropriate information for the priority change is included in the Markov process describing congestion.

An alternative terminology for describing the effects of events is the concept of a routing process. This was the approach of J.R. Jackson [4] in his pioneering paper on queueing networks. He assumed that there was a queue associated with each of the servers in the system. When a server completed a task he started on the next job in his queue. The job which he had just finished went to any of the queues or left the system according to probabilities which depended only on which server had finished work on the job. Arriving jobs went to their first queue according to a fixed set of probabilities. Jackson called this Markov routing. The models considered here will retain the Markov property but allow the probabilities of transfers to depend on the service time of the service just completed or more generally on the elapsed

times of all intervals in process. In addition, it is possible to have one or more servers serving one or more queues. In some cases several servers may work on the same job simultaneously. In addition whether a free server begins work or not will be allowed to depend on the current level of congestion in the system. Thus a server who has a job waiting for his attention may remain idle because of excessive congestion elsewhere in the system.

Congestion Process

The congestion process to be developed here measures the number of jobs in k different catagories in the system in continuous time. Thus it is a k dimensional vector valued stochastic process N(t) with non negative interger coordinates. Since changes only occur when events occur it can be assumed that N(t) is continuous from the right i.e. limit $N(\tau) = N(t)$ when τ decreases to t. This process is generally not Markovian but becomes Markovian if a k* dimensional vector valued supplementary variable W(t) is added. The coordinates of W(t) measure the elapsed time of the intervals in process. The set of values of N and W may be restricted in any coordinate to a finite range. As long as arrivals and service operations involve single jobs, then, if there can be i jobs of type j in the system at a given time, at some previous time there must have been i-1 jobs of type j in the system. Similarly if w is a possible time for the elapsed interval of type j then all smaller times must have occurred. Thus in both cases any restriction of the coordinates of N, W must be to an interval.

When N(t) = n assume that there is a set S_n of the integers [1,k*] which identifies the types of intervals which are in process. Let e_{S_n} be a vector with a 1 in coordinate k if $k \in S_n$ and a 0 if $k \notin S_n$. If no interval ends in Δt then $W(t+\Delta t) = W(t) + \Delta t\, e_{S_n}$. With probability $h_i(w)\,\Delta t$ an interval of type i terminates in $(t,t+\Delta t)$. If N(t), W(t) = n,w, there is a transition from n,w to n´,w´, according to the routing probabilities $\mu_i(n',w'|n,w)$. It is

assumed that there are a finite number of these conditional probabilities and that they sum to 1 and that they are continuous from the right in w. Since the routing process changes the location or classification of a single job all these transitions are of the nearest neighbor variety. Thus $\mu_i(n',w'|n, w)$ is zero for all n' for which $|n'-n| > 1$ where the norm is the maximum absolute value of the coordinates. Again $w' = w + \Delta t\, e_{S_n}$ although with some exceptions. First the i th coordinate of w' must be zero since this type of interval ended. Other exceptions occur when there is a premption or there is a cancellation. In this case some coordinate w_j', $j \neq i$ will be either w_j or 0. This defines the infinitesimal generator and therefore the transition function of the congestion process[2].

Transition Probabilities

More interesting than the formula for the transition probabilities in terms of the infinitesimal generator is an alternative which explicitly uses the event structure. The transition probabilities can be represented in terms of two simple operators. The first is the no event operator $T^o(t)$ defined by

$$T^o(t)(n,w,w') = \begin{cases} \prod_{i\in S_n} \dfrac{1 - F_i(t+w_i)}{1 - F_i(w_i)} & \text{if } w_i' = t + w_i \quad i\in S_n \\ & \quad w_i' = w_i \quad i\notin S_n \\ 0 & \text{otherwise} \end{cases}$$

The second is the one event operator $T^1(t)$ defined by

$$T^1(t)(n,w,n',w') = \sum_{i\in S_n} \mu_i(n',w',n,w+te_{S_n}) \frac{f_i(t+w_i)dt}{1 - F(w_i)} \prod_{\substack{j\in S_n \\ j\neq i}} \frac{1 - F_j(t+w_j)}{1 - F_j(w_j)}$$

This function is the probability of a transition from n,w at 0 to n´,w´ at some time in the interval t,t+dt. Using the continuity from the right of the routing probabilities, this can also be considered the probability of a transition from N(0) = n, W(0) = w to N(t) = n´,W(t)ε(w´,w´+dte$_{S_{n'}}$). In addition, the elapsed time of any interrupted intervals is specified only to an accuracy of dt.

The probability of a two event transition n,w to a state n´,w´ with the second event occuring at t is

$$T^2(t)(n,w,n',w') = \int_0^t \int_{w^*} \sum_{n^*} T^1(s)(n,w,n^*,w^*) T^1(t-s)(n^*,w^*,n',w')$$

The sum only involves a finite number of terms corresponding to possible events at time s. The intergrals are somewhat deceptive because not all values of w* have positive probability. To make a positive contribution w* must satisfy $w' = w^* + (t-s)\, e_{S_{n^*}}$ except for the coordinates which become 0 in the transition from n* tc n´. The result may be only the sum of a finite number of products of terms, each of which is proportional to dt. The appropriate interpretation here uses both the approximation in space and time. Thus the result will be considered proprotional to dt times the product of infinitessimals dw_i for those values of i for which intervals either start or are interrupted at times s. On the other hand, the event at s may lead to a new state n* in which no intervals are interrupted and no new intervals are begun. In this case $S_{n^*} \subset S_n$ and the intergral includes non zero terms for all s in (o,t).

The definition of $T^2(t)$ can be used recursively to define

$$T^k(t)(n,w,n',w') = \int_0^t \int_{w^*} \sum_{n^*} T^{k-1}(s)(n,w,n^*,w^*) T^1(t-s)(n^*,w^*,n',w')$$

Eventually all of the initial intervals terminate and $T^k(t)(n,w,n',w')$ becomes a function proportional to dt times $\pi\, dw_i$ for $i \in S_{n'}$ or the i th type interval is suspended in n'. To simplify later expressions the integrations over the state variables will be symbolized by $T^{k-l}(s) * T^l(t-\Delta)$ so that

$$T^k(t) = \int_o^t T^{k-l}(s) * T^l(t-s)$$

Now define

$$H^o(t)\ (n,w,n',w') = T^o(t)(n,w.w')$$

$$H^k(t)(n,w,n',w') = \int_o^t \int_{w^*} T^k(s)(n,w,n',w^*)\ T^o(t-s)(n',w^*,w')$$

In the more concise notation

$$H^k(t) = \int_o^t T^k(s) * T^o(t-s)$$

This family of functions give the probability of a k event transition from n,w at o to n' and approximately w' at time t.

Since with probability 1 only a finite number of events occur in a finite time the probability of a transition from n, w at o to n' and approximately w' at t must be

$$\sum_{k=0}^{\infty} H^k(t)$$

This is a conservative transition function i.e. the sum over all states at time t is 1. It satisfies the differential equation defined by the infinitessimal generator.

Example

Before justifying the last two statements, consider the process of this type for an M/M/1 queue. The state is a single non negative integer variable and two supplementary variables w_1 and w_2 are needed for elapsed interarrival and service times respectively.

$$T^{0}(t)(n,w,w') = \begin{cases} e^{-\lambda t} & n=0 \quad w_1 \geq 0 \quad w_2=0 \\ & \quad w_1'=w_1+t \quad w_2'=0 \\ e^{-(\lambda+\mu)t} & n>0 \quad w_1 \geq 0 \quad w_2 \geq 0 \\ & \quad w_1'=w_1+t \quad w_2'=w_2+t \\ 0 & \text{otherwise} \end{cases}$$

$$T^{1}(t)(n,w,n',w') = \begin{cases} \lambda e^{-\lambda t}dt & n=0,\ w_1 \geq 0,\ w_2=0 \\ & w=1,\ w_1'=0,\ w_2'=0 \\ \lambda e^{-(\lambda+\mu)t}dt & n>0,\ w_1 \geq 0,\ w_2 \geq 0 \\ & n'=n+1 \quad w'=0 \quad w_2'=w_2+t \\ \mu e^{-(\lambda+\mu)t}dt & n>0 \quad w_1 \geq 0 \quad w_2 \geq 0 \\ & n'=n-1 \quad w_1'=w_1+t \quad w_2'=0 \\ 0 & \text{otherwise} \end{cases}$$

In space terms

$$T^1(t)(n,w,n',w') = \begin{cases} \lambda e^{-\lambda t}\, dw_1 & n=0 \quad w_1 \geq 0 \quad w_2=0 \\ & n'=1 \quad w_1'=0 \quad w_2'=0 \\ \lambda e^{-(\lambda+\mu)t}\, dw_1 & n>0 \quad w_1 \geq 0 \quad w_2 \geq 0 \\ & n'=n+1 \quad w_1'=0 \quad w_2'=w_2+t \\ \mu e^{-(\lambda+\mu)t}\, dw_2 & n>0 \quad w_1 \geq 0 \quad w_2 \geq 0 \\ & n'=n-1 \quad w_1'=w_1+t \quad w_2'=0 \\ 0 & \text{otherwise} \end{cases}$$

The special case of n=1, n'=o and the termination of a service event produces exactly $w_2'=o$ not w_2' in o, dw_2' since a new service interval is not started.

Next

$$T^2(t)(n,w,n',w') = \begin{cases} \lambda\mu e^{-\lambda t} e^{-\mu w_1'}\, dt\, dw_1' & n=0 \quad w_1 \geq 0 \quad w_2=0 \\ & n'=0 \quad w_1'=w_1+t \quad w_2'=0 \\ \lambda^2 e^{-\lambda t} e^{-\mu w_2'}\, dt\, dw_2' & n=0 \quad w_1 \geq 0 \quad w_2=0 \\ & n'=2 \quad w_1'=0 \quad t \geq w_2' \geq 0 \\ \lambda e^{-\lambda t}(1-e^{-\mu t})\, dt & n=1 \quad w_1 \geq 0 \quad w_2 \geq 0 \\ & n'=1 \quad w_1'=0 \quad w_2'=0 \\ \lambda\mu e^{-(\lambda+\mu)t}\, dt\, dw_1' & n \geq 1 \quad w_1 \geq 0 \quad w_2 \geq 0 \\ & n'=n \quad t \geq w_1' \geq 0 \quad w_2'=0 \\ \lambda^2 t e^{-(\lambda+\mu)t}\, dt & n \geq 1 \quad w_1 \geq 0 \quad w_2 \geq 0 \\ & n'=n+2 \quad w_1'=0 \quad w_2'=w_2+t \end{cases}$$

$$
\begin{cases}
\mu^2 te^{-(\lambda+\mu)t}dt & n\geq 2 \quad w_1\geq 0 \quad w_2\geq 0 \\
 & n'=n-2 \quad w_1'=w_1+t \quad w_2'=0 \\
\lambda\mu e^{-(\lambda+\mu)t}dtdw_2' & n\geq 2 \quad w_1\geq 0 \quad w_2\geq 0 \\
 & n'=n \quad w_1'=0 \quad t\geq w_2'\geq 0
\end{cases}
$$

The transitions which terminate intervals and start new ones of the same type produce new values of the supplementary variable between o and an infinitessimal even though the probability is not proportional to this infinitessimal. In addition for $n=n'$ in the general case, two expressions overlap at $w_1'=w_2'=o$ and the appropriate probability is the sum of the two terms.

It is interesting to interpret $T^2(t)$ for $n\geq 2$ with the elementary result that the probability that the second of two events ends in t,t+dt. Regardless of the initial value of w this is

$$(\lambda+\mu)^2 te^{-(\lambda+\mu)t}dt$$

The time t must be the sum of two independent negative exponentially distributed intervals each with mean $1/(\lambda+\mu)$ because in turn each is the minimum of two independent exponential variables with means $1/\lambda$ and $1/\mu$. To see that $T^2(t)$ merely distributes this probability first the transitions to $n'=n$, $w'\geq 0$, $w_2'=0$ and $n'=n$, $w_1'=0$, $w_2'\geq 0$ must be intergrated with respect to w_1' and w_2' respectively over the intervals (o,t). This provides the missing t values in these terms and together with the two other possibilities $n'=n-2$ and $n'=n+2$ the sum is the probability of two events.

Interpreting the entries of $T^k(t)$ as merely refinements of the Poisson counting process is possible for all states for which $n\geq k$. Consider $T^3(t)$ for $n\geq 3$

$$T^3(t) = \begin{cases} (\mu^3 t^2 e^{-(\lambda+\mu)t}\,dt)/2 & n\geq 3 \quad w_1\geq 0 \quad w_2\geq 0 \\ & n'=n-3 \quad w_1'=w_1+t \quad w_2'=0 \\ \lambda\mu^2 t e^{-(\lambda+\mu)t}\,dt\,dw_1' & n\geq 3 \quad w_1\geq 0 \quad w_2\geq 0 \\ & n'=n-1 \quad t\geq w_1'\geq 0 \quad w_2'=0 \\ \lambda\mu^2 (t-w_2') e^{-(\lambda+\mu)t}\,dt\,dw_2' & n\geq 3 \quad w_1\geq 0 \quad w_2\geq 0 \\ & n'=n-1 \quad w_1'=0 \quad t\geq w_2'\geq 0 \\ \lambda\mu^2 (t-w_1') e^{-(\lambda+\mu)t}\,dt\,dw_1' & n\geq 3 \quad w_1\geq 0 \quad w_2\geq 0 \\ & n'=n+1 \quad t\geq w_1'\geq 0 \quad w_2'=0 \\ \lambda\mu^2 t e^{-(\lambda+\mu)t}\,dt\,dw_2' & n\geq 3 \quad w_1\geq 0 \quad w_2\geq 0 \\ & n'=n+1 \quad w_1'=0 \quad t\geq w_2'\geq 0 \\ (\lambda^3 t^2 e^{-(\lambda+\mu)t}\,dt)/2 & n\geq 3 \quad w_1\geq 0 \quad w_2\geq 0 \\ & n'=n+3 \quad w_1'=0 \quad w_2'=w_2+t \end{cases}$$

The calculations here are slightly more involved than before. For example, for $n'=n-1$, $t\geq w_1'\geq 0$ $w_2'=0$ it is necessary to simplify

$$\int_{w_2^*=0}^{t-w_1'} \lambda\mu e^{-(\lambda+\mu)(t-w_1')}\,dt\; dw_2^* \mu e^{-(\lambda+\mu)w_1'} dw_1'$$

$$+ \int_{w_1^*=0}^{w_1'} \lambda\mu e^{-(\lambda+\mu)(t-w_1'+w_2^*)}\,dt\; dw_1^* \mu e^{-(\lambda+\mu)(w_1'-w_1^*)} dw_1'$$

Again, integrating the supplementary variables and adding produces

$$\frac{(\lambda+\mu)^3 t^2 e^{-(\lambda+\mu)t}\,dt}{2}$$

Convergence

If the integrals defining the functions H^k are integrated by parts, the results are

$$H^k(t) = \int_o^t F^k(\xi) - \int_o^t \int_o^s T^k(\xi) * (-T^{o\prime}(s-\xi)ds)$$

The prime on the functions indicates differentiation with respect to the time arguement and is justifiable from the definition of T^o. The function H^o can be given a similar form as

$$H^o(t) = I - \int_o^t (-T^{o\prime}(\xi)d\xi)$$

where I is the identity function $I(n,w,n',w')=1$ if $n=n'$ and $w=w'$ and 0 otherwise These first of the integrals on the right is the joint probabilities of k or more events in (o,t) and the state at the k th event. The second integral is the probability of k+1 or more events in (o,t) and the state of the k th event. From their definitions, the function $-T^{o\prime}$ is equal to the sum over the states to which a transition leads of the function T^1. If these terms are summed over the possible states at the time of the k th event, all terms but I cancel in ΣH^k. The error in the sum of the first m terms is the probability of m+1 or mor events in (o,t). This term must go to 0 for all starting states as m goes to infinity as discussed earlier. Thus ΣH^k converges to a function which sums to 1 for each initial condition.

Differential Equation

Notice that the infinitessimal generator is obtained from an operator A(dt) describing transitions in dt. It can be given a representation as a function $A(n,w,n',w')dt$ and

$$T^o * Adt = T^1(t+dt) + T^o(t+dt)$$

The probabilities of no events produce $\pi^0(t+\Delta t)$ since

$$\prod_{j\in S_n} \left(\frac{1-F_j(t+dt+w_j)}{1-F_j(w_j)}\right) = \prod_{j\in S_n} \left(\frac{1-F_j(t+w_j) - f_j(t+w_j)dt}{1-F_j(w_j)}\right) + o(dt)$$

$$= \prod_{j\in S_n} \frac{1-F_j(t+w_j)}{1-F_j(w_j)} - \sum_{i\in S_n} \prod_{\substack{i\in S_n \\ i\neq j}} \frac{1-F_i(t+w_i)}{1-F_i(w_i)} \sum_{j\in S_n} \frac{f_j(t+w_j)dt}{1-F_j(t+w_j)}) + o(dt)$$

Similarly for small dt to an accuracy of o(dt)

$$\mu(n',w',n,w+(t+dt)e_{S_n})\, h_j(t+dt+w_j)dt \prod_{\substack{i\in S_n \\ i\neq j}} \left(\frac{1-F_i(t+dt+w_i)}{1-F_i(w_i)}\right) =$$

$$\mu(n',w',n,w+te_{S_n}) \frac{f_j(t+w_j)}{1-F_j(t+w_j)}dt \prod_{\substack{i\in S_n \\ i\neq j}} \left(\frac{1-F_i(t+w_i)}{1-F_i(w_i)}\right) + o(dt)$$

because of the assumption of the continuity of the interval densities as well as the continuity from the right of the routing probabilities. Applying A dt to $H^k(t)$ gives

$$H^k(t) * Adt = [\int_0^t T^k(s) * T^0(t-s)] * Adt$$

$$= \int_0^t T^k(s) * [T^0(t+dt-s) + T^1(t+dt-s)]$$

$$= H^k(t+dt) - T^k(t+dt) + T^{k+1}(t+dt) + o(dt)$$

The error term comes from $-T^k(t+dt) * T^1(o)$ which is of order dt^2. Thus the sums satisfy

$$\sum_{k=0}^{\infty} H^k(t)\, Adt = \sum_{k=0}^{\infty} H^k(t+dt) + o(dt)$$

Thus these functions satisfy the differential equation of the transition probabilities.

$$\lim_{dt\to 0} \sum_{k=0}^{\infty} \frac{H^k(t) * (Adt - I)}{dt} = \lim_{dt\to 0} \frac{\sum_{k=0}^{\infty} H^k(t+dt) - \sum_{k=0}^{\infty} H^k(t)}{dt}$$

Events Beginning Subintervals

The development of the k event transition functions was based on events terminating subintervals. This requires the addition of a final subinterval which contains no event. This inclusion makes the process symetric in that one can proceed backward in time from time t to the k th event then to the k-1 st, etc. to the 1st and then to time 0 through an interval containing no events. The theoretical difficulty of the backward movement is the possibility that there is no last event in (o,t) and thus no 1st event for the backward process [1]. Here there is no difficulty since there are only a finite number of events in any finite interval with probability 1.

The shift to the backward approach is based on simple properties of T^o and T^1 which are immediately verifyable from their definitions. First

$$T^1(t) = T^o(t) * T^1(o)$$

Thus

$$\begin{aligned} H^k(t) &= \int_o^t H^{k-1}(s) * T^o(t-s) \\ &= \int_o^t \left[\int_o^s T^{k-1}(\xi) * T^1(s-\xi)\right] * T^o(t-s) \\ &= \int_o^t \left[\int_o^s H^{k-1}(\xi) * T^o(s-\xi) * T^1(o)\right] * T^o(t-s) \\ &= \int_o^t H^{k-1}(s) * T^1(o) * T^o(t-s) \end{aligned}$$

Introducing a new function $T^*(t) = T^1(0)T^o(t)$ this gives

$$H^k = \int_o^t H^{k-1}(s) * T^*(t-s)$$

The value of function $T^*(\tau)$ at (n,w,n',w') is the probability of a transition from $n,w,$ to n',w' at time 0 followed by an interval of length τ which contains no event.

Discrete Time Processes

The calculational problems of this recursive structure are obvious as is one possible simplification. The Laplace transform will eliminate the convolution operation. Let

$$\tilde{T}^k(\alpha) = \int_0^\infty e^{-\alpha t} T^k(t)$$

$$\tilde{H}^k(\alpha) = \int_0^\infty e^{-\alpha t} H^k(t)dt$$

In Laplace transform terms the recursive equation becomes

$$\tilde{T}^k = \tilde{T}^{k-1} * \tilde{T}^1$$

$$\tilde{H}^k = \tilde{T}^k * \tilde{T}^o$$

Since it is the sum of the H^k which is required it is perhaps better to consider

$$\tilde{S}^1 = \tilde{T}^1$$

$$\tilde{S}^k = \tilde{S}^k * \tilde{T}^1 + \tilde{T}^1$$

The limit then satisfies

$$\tilde{S}^\infty * (I-\tilde{T}^1) = \tilde{T}^1$$

or

$$\tilde{S}^\infty = \tilde{T}^1(I-\tilde{T}^1)^{-1}$$

where I is the identity operator.

The existance of the inverse can be justified directly because $|\tilde{T}^1| < 1$ [3, p. 52] For this one uses the total variation as the vector norm and then the associated operator norm. This means that in computing the norm of $\tilde{T}^1$ it is only necessary

to consider $V * \tilde{T}^{-1}$ for any probability measure V. Only $\alpha > 0$ is of interest. For α real, $\tilde{T}^{-1}(\alpha)(n,w,n',w')$ is the probability of a transition from $n,w,$ to n', to w' before an exponential interval with mean $1/\alpha$ terminates. Since such a departure is not certain, $V * \tilde{T}^{-1}$ is a non negative measure which does not sum to 1.

The probability interpretation of the Laplace transform can be made complete by considering $\tilde{T}^{-1}(\alpha)$ as the single step operator of a discrete time Markov process. The process is not conservative since at each stage there is positive probability that the exponential interval terminates before an event which changes the value of N. The limiting value $\alpha=0$ corresponds to the original congestion process observed just after the occurance times of the events. Although these processes are not chains in the discrete variable as in semi-Markov processes, they do provide a major simplification.

1. Chung, K.L., Markov Chains, 2nd Ed., Springer-Verlag, 1967.

2. Dynkin, E.B., Markov Processes 1, Springer-Verlag. 1965.

3. Halmos, P.R., Introduction to Hilbert Space, Chlesea, 1951.

4. Jackson, J.R., "Jobshop-Like Queueing Systems", Mgt. Sci., Vol. 10, 1963, pp. 131-142.

UNIVERSITY OF ILLINOIS-URBANA
3 0112 060296586